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SURF BEAT 'SHOALING'

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Abstract

This paper makes use of the complete, solution for transient, long waves forced by short-wave groups at constant depth to provide an intuitive understanding of several features of the behavior of real surf-beat on real beaches namely that:

1. The long waves are generally not in exact counter-phase with the short-wave envelopes as in the steady solution by Longuet-Higgins & Stewart (1962), hereafter L-H&S. Rather, the bottom of the long-wave trough is delayed relative to the short-wave envelope peak.
2. The long wave generated by a single shortwave pulse consists of a leading positive surge as well as the L-H&S depression.
3. Long waves of higher frequency are observed to 'shoal' faster than lower frequency ones.

Transient behavior of surf-beat and other forced long waves is complicated by the fact that growth from a given shape cannot generally happen by simple up-scaling of the pre-existing shape. For example, simple up-scaling of the L-H&S depression under a single wave pulse would violate conservation of volume. The extra water needs to be put somewhere within a finite distance. The complete, transient solution to the linear, constant depth problem shows how this is achieved. The qualitative process is adaptable to shoaling scenarios.

Key words: surf beat, forced long waves, resonance, wave groups

1. Introduction

It has been understood since the early 1960s that, waves exert radiation stress or wave thrust, proportional to the wave height squared (S_{xx}~H^2), which drives inshore phenomena like wave setup and long-shore currents. In deeper water, the effect of variable S_{xx} is analogous to the effect of a variable air pressure on the surface. Thus, S_{xx}~H^2 drives water from areas with big waves towards areas with smaller waves, forming long waves, which are coherent with the wave groups and follows them in the steady state.

The long wave surface elevation $\eta_L(x,t)$ must, at constant depth h, satisfy

$$\frac{\partial^2 \eta_L}{\partial t^2} - gh\frac{\partial^2 \eta_L}{\partial x^2} = \frac{1}{\rho}\frac{\partial^2 S_{xx}}{\partial x^2} \quad (1)$$

where g is the acceleration due to gravity an ρ is the density of water.

This equation has steady solutions corresponding to steady wave groups with radiation stress of the form

$$S_{xx}(x,t) = S_0\, \mathrm{f}(x\text{-}c_g t) \quad (2)$$

ie, with general magnitude S_0, and travelling with constant shape given by the function f at the group velocity c_g. The steady solution is, as per L-H&S:

$$\eta_B(x,t) = a_B \mathrm{f}(x - c_g t) = \frac{-S_o/\rho g h}{gh - c_g^2}\, \mathrm{f}(x - c_g t) + \text{Constant} \quad (3)$$

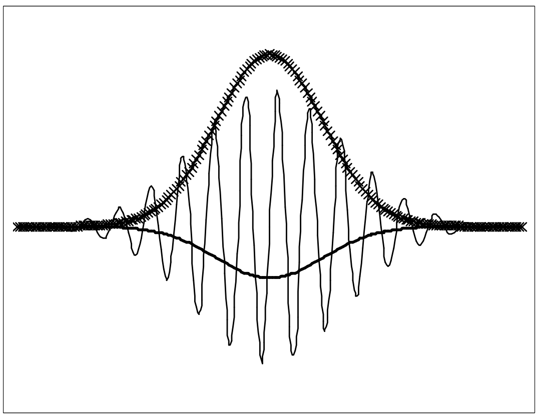

Figure 1:
The L-H&S solution (3) for a single wave pulse is a depression centred under the forcing envelope.

Here the subscript B indicates that this is a Bound wave, which travels with the short-wave group and it has the shape of the (inverted) short-wave envelope (~ -S_0f).

This solution has been widely used to interpret observations of long waves associated with short-wave groups even when the assumptions behind this solution are not fulfilled, eg, in case of one or more of the following

1. The depth is not constant
2. The shortwave envelope changes shape due to dispersion
3. A steady state has not yet been reached

The following describes and explains the ensuing discrepancies, which can be understood, at least qualitatively, through the complete, linear solution for constant depth.

2. The complete solution for forced long waves at constant depth

The complete solution includes forward and backward free waves as well as the bound wave given by (3). Ie, the complete solution has the form

$$\eta_L(x,t) = \eta_B(x-c_g t) + \eta_{free}^{forward}(x-\sqrt{gh}\,t) + \eta_{free}^{backward}(x+\sqrt{gh}\,t) \quad (4)$$

of which the steady solution (3) is the first term on the right.

Because the bound and free waves propagate with different speeds, the complete picture evolves in in a fairly complicated-looking fashion, although it is a simple superposition of three waves. Several examples were discussed in detail by Nielsen et al (2008) and Nielsen (2009), Section 1.3.3.4.

Those studies were prompted by an alert student asking: ”So, L-H&S and others give steady state solutions for various types of forced long waves, storm surges, surf-beat tsunami and.. . But, how do we get from an initially flat ocean at rest to this fully developed steady state?”

For the scenario where the forcing, e g (2) is switched on at time zero and then remains steady, the development is as follows:

The initially flat ocean corresponds to the three waves cancelling each other:

$$\eta_L = \eta_B + \eta_{free}^{forward} + \eta_{free}^{backward} \equiv 0 \qquad \text{at } t=0 \quad (5)$$

This condition can be satisfied if the free waves have the same shape, f, as the forcing. The relative magnitudes of the forward and backward free waves are found by considering overall momentum. For the case of starting with an initially quiescent ocean and correspondingly zero overall momentum one finds

$$(a_{free}^{forward}, a_{free}^{backward}) = \left(-\frac{a_B}{2}[1+\frac{c_g}{\sqrt{gh}}],\ -\frac{a_B}{2}[1-\frac{c_g}{\sqrt{gh}}] \right) \quad (6)$$

The steady state is achieved, when the free waves have separated completely from the bound wave due to their different speed: $c_g < \sqrt{gh}$. The backward free wave leaves the scene quickly because of its greater relative speed $-\sqrt{gh}-c_g$ versus $\sqrt{gh}-c_g$. After that the process in Figure 2 plays out.

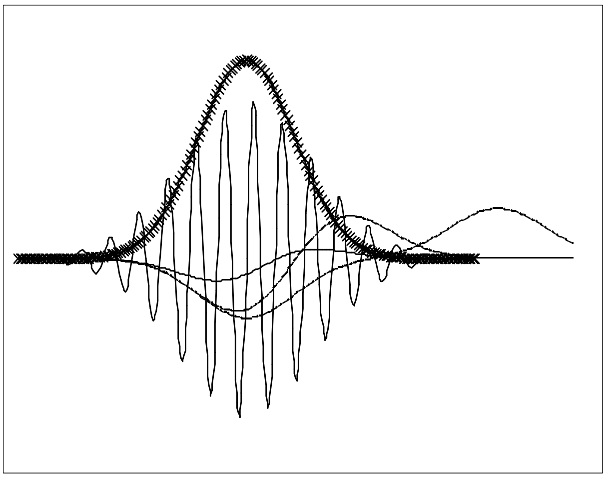

Figure 2:
The sum: $\eta_B(x,t)+\eta_{free}^{forward}(x,t)$, thin lines, at different stages after the switch-on of the forcing (×).
Eventually $\eta_{free}^{forward}(x,t)$ disappears out of the picture towards the right and the steady state represented by (3) alone is reached as in Figure 1.

The scenario in Figure 2 has been realised by Baldock (pers com 2016) in the laboratory. He generated a single short-wave pulse and observed that, as the pulse propagated down the flume, the depression and the positive surge in front developed. This corresponds to the simple observation that, when a wave maker starts doing simple harmonic motion in a flume with water at rest, an observer at the 'beach end' will initially see the waterline move up quasi-steadily. This is due to $\eta_{\text{free}}^{\text{forward}}$ arriving at the beach ahead of the short-wave train.

2.1 ***What happens at resonance?***

The special behaviour at resonance is important because, surf-beat on a sloping beach, although not quite resonant, behaves much like the resonant, constant depth solution, which we shall therefore now develop.

The steady solution (3) blows up for $c_{\text{g}} \to \sqrt{gh}$ so, there is no steady solution for $c_{\text{g}} = \sqrt{gh}$. What happens then is that, the bound wave and the forward free wave, who now have the same speed, merge into a wave of the form $t \times \mathrm{f}'(x - \sqrt{gh}\,t)$, ie, growing linearly with time and in the shape of the derivative of the forcing shape. This is analogous to the well known resonant solution for the displacement of a mass on a spring forced by $F_0 \sin \omega_0 t$. It grows as $t \cos \omega_0 t$, where the cos-function is the derivative of the sin-function. For surf beat the derivation goes as follows:

We note that the backward free wave vanishes at resonance according to (6). Then, using the linear wave theory expression for the group velocity:

$$c_{\text{g}} = \sqrt{gh}\left[1 - \frac{1}{2}k_0 h + O(k_0 h)^2\right] \tag{7}$$

where $k_0 = 4\pi^2/gT^2$ is the deep water wave number, one finds, in the scenario where the forcing starts and both of η_{B} and $\eta_{\text{free}}^{\text{forward}}$ are created at t=0:

$$\begin{aligned}
\eta_{\text{L}}(x,t) = \eta_{\text{B}} + \eta_{\text{free}}^{\text{forward}} &= \frac{-S_o/\rho gh}{1 - c_{\text{g}}^2/gh}\left(\mathrm{f}(x - c_{\text{g}}t) - \frac{1}{2}[1 + \frac{c_{\text{g}}}{\sqrt{gh}}]\mathrm{f}(x - \sqrt{gh}\,t)\right) \\
&\xrightarrow[k_o h \to o]{} \frac{-S_0/\rho gh}{k_0 h}\left(\mathrm{f}(x - \sqrt{gh}[1 - \frac{k_0 h}{2}]t) - \mathrm{f}(x - \sqrt{gh}\,t)\right) \\
&\xrightarrow[k_o h \to o]{} \frac{-S_0/\rho gh}{k_0 h}\left(\sqrt{gh}\frac{k_0 h}{2} t\,\mathrm{f}'(x - \sqrt{gh}\,t)\right) \\
&= \frac{-S_0}{2\rho\sqrt{gh}}\; t\;\mathrm{f}'(x - \sqrt{gh}\,t)
\end{aligned} \tag{8}$$

For example, this means that a single wave pulse with say a Gaussian–shaped envelope, which generates a Gaussian-shaped depression in the steady state, will at resonance generate an И-wave. The quasi sinusoidal long waves under bi-cromatic wave groups will be shifted a quarter of a long-wave wave length at resonance corresponding to a phase shift of $\pi/2$ for harmonic components of the long wave.

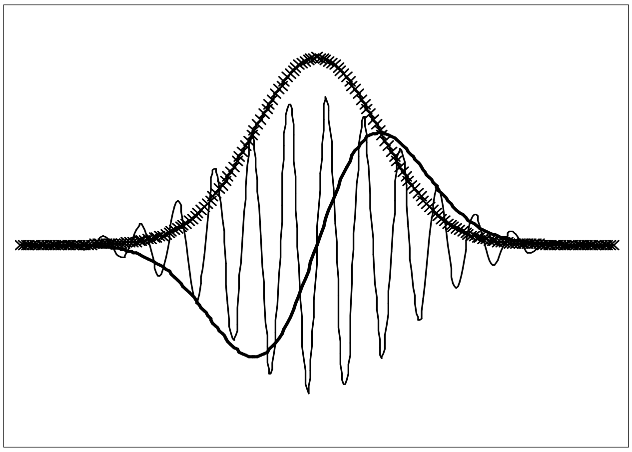

Figure 3:
At resonance, $c_g = \sqrt{gh}$, which happens in the shallow water limit, the Gaussian shaped S_{xx} forcing $-\times-$ from the wave pulse generates an И-shaped long wave, which grows linearly with time.

3. Long wave development on a slope

For a short-wave pulse, which propagates up a slope the forcing $S_{xx}(x,t)$ increases in strength due to short-wave shoaling and the response strengthens as the denominator in (3) becomes smaller. This process corresponds, in the language of the transient solution (4), to the bound wave growing deeper, while successive incremental forward free waves are generated, to accommodate the extra volume, and propagate ahead with relative speed $\sqrt{gh}-c_g$. As this relative velocity becomes smaller in shallower and shallower water, in accordance with (7), the free wave does not get away, and the shape of the complete solution approaches the qualitative form of the resonant constant-depth solution in Figure 3.

This has been borne out by the detailed laboratory measurements and RANS numerical calculations of Lara et al (2011).

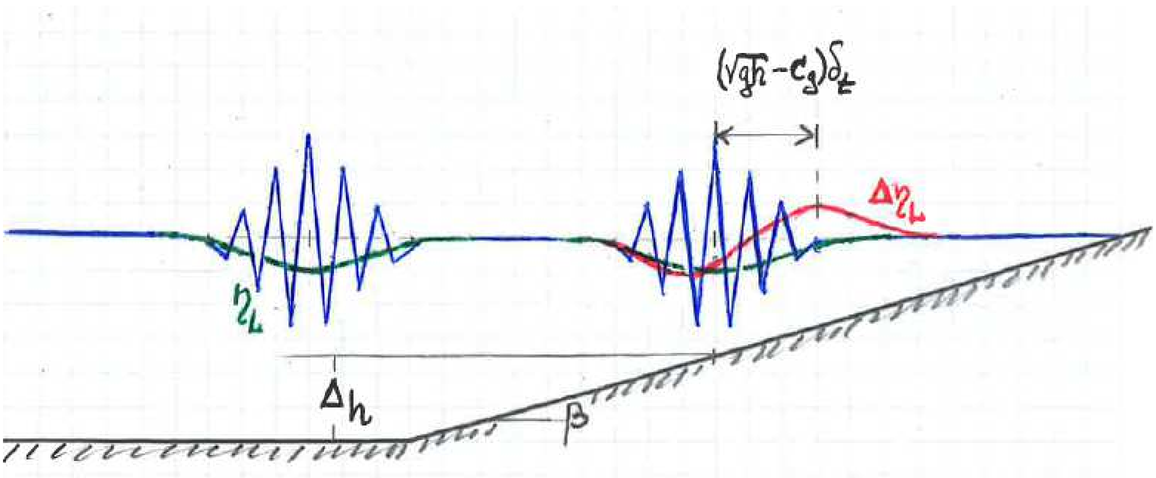

Figure 5: Long wave incremented by $\Delta\eta_L$ due to depth decrease Δ_h. The forward moving free wave has advanced $(\sqrt{gh}-c_g)\delta_t$ relative to the short-wave group and the pre-existing long wave. The increment $\Delta\eta_L$ is exaggerated compared to the pre-existing long wave for illustrative purposes. Importantly, we note that the incremental long wave is volume neutral as required.

In terms of the complete linear constant-depth solution (4) the incremental long wave consists of a negative, incremental bound wave and an initially cancelling, incremental forward free wave and can be approximated as follows:

$$\Delta\eta_L \approx \Delta\eta_B + \Delta\eta_{\text{free}}^{\text{forward}} = \Delta a_B\left(\mathrm{f}(x-c_g\delta_t) - \frac{1}{2}[1+\frac{c_g}{\sqrt{gh}}]\mathrm{f}(x-\sqrt{gh}\,\delta_t)\right)$$
$$\xrightarrow[k_o h\to o]{} \Delta a_B\left(\mathrm{f}(x-\sqrt{gh}[1-\frac{k_0 h}{2}]\delta_t) - \mathrm{f}(x-\sqrt{gh}\,\delta_t)\right) \tag{9}$$
$$\xrightarrow[k_o h\to o]{} \Delta a_B\sqrt{gh}\frac{k_0 h}{2}\delta_t\,\mathrm{f}'$$

and with $\delta_t = \dfrac{\Delta_h}{\beta\sqrt{gh}}$

$$\Delta\eta_L = \Delta\eta_B\frac{k_0 h}{2}\frac{\Delta_h}{\beta}\mathrm{f}' \tag{10}$$

While the pressure gradients associated with the radiation stress of the short waves do no work on the steady, pre-existing long wave, they can do work on the incrememt $\Delta\eta_L$ as discussed in section 5.

4. L-H&S do not conserve mass in a shoaling situation

Perhaps the most obvious indication that more than the steady L-H&S solution is needed in relation to coastal processes is that this steady solution will not conserve mass, if applied in a quasi-steady fashion to a shoaling scenario.

This point, is easily overlooked when only periodic, bicromatic wave groups are considered, because, they generate sinuous long waves, where mass can be balanced between neighbouring positive and negative parts. Short, isolated wave groups, on the other hand, as studied by Watson et al (1994) and later by Baldock (2006) bring this point out nicely. That is, for a single, isolated wave pulse the steady solution is a single depression as per Figure 1. If the same wave group is transferred to a shallower depth, the steady solution (3) says that the depression must be deeper because the denominator $gh\text{-}c_g^2$ gets smaller. However, there is nowhere to put the excess water. Accommodating it in the constant in (3) is not a physical option because, this would involve redistributing water all the way out to +/- infinity in a finite time. Water can at most travel at speed $\sqrt{gh}$.

The complete solution on the other hand enables conservation of mass during a quasi-steady shoaling process as illustrated in Figure 5.

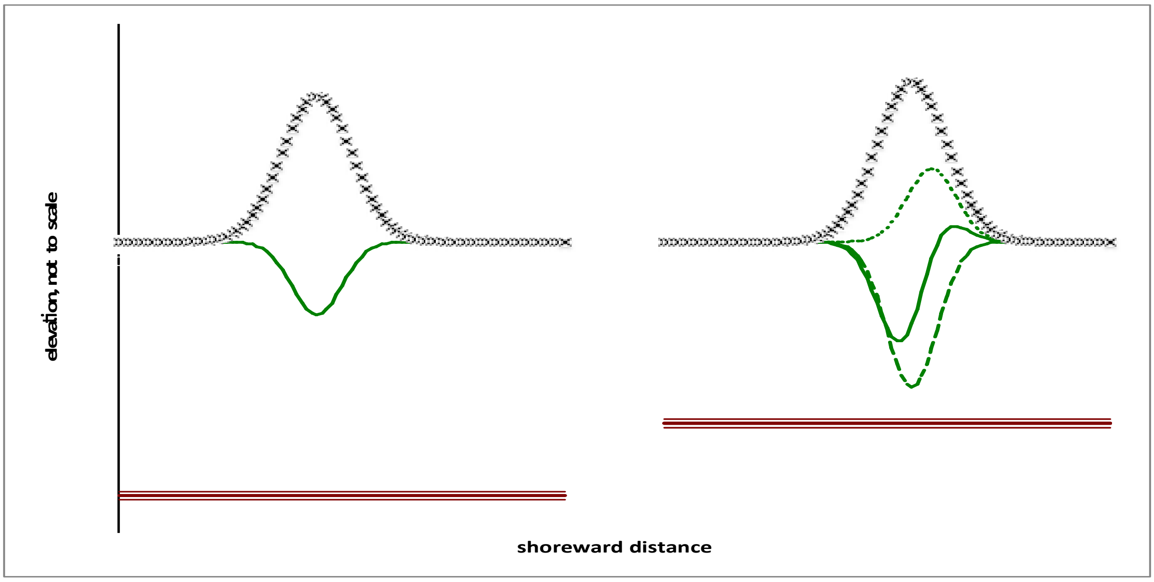

Figure 5:
Left: Short wave pulse with $S_{xx}(x,t)$ indicated by (×××) and the associated steady, forced long wave with the same shape (inverted) in accordance with (3).
Right: Same pulse, somehow progressed to a shallower depth. The expression (3) has now deepened in accordance to L-H&S's $h^{-5/2}$ law **(- - -)**. To accommodate the extra water a positive, forward-moving, free wave has formed (••••). The total long wave given by (4) (___) now has a leading positive part as well as a negative part, now not quite centred under $S_{xx}(x,t)$, but lagging.

5 Long wave growth requires a lag, which is absent in the L-H&S solution

The right-hand part of Figure 5, now offers insight into how the short-wave radiation stress gradient can do work on the long wave: That is, the complete long wave (___) is now lagging $S_{xx}(x,t)$, which corresponds to a phase-lag of harmonic components as observed by Elgar et al (1992) and Janssen et al (2003).

The radiation stress gradients do work on the long wave via the long wave velocities U_L, as per

$$\dot{E} = \int \frac{\partial S_{xx}}{\partial x} U_L \, dx = \int \frac{\partial S_{xx}}{\partial x} \frac{c_g}{h} \eta_L \, dx \tag{11}$$

which, under steady (constant depth) conditions, with η_L given by (3) is zero as per

$$\int \frac{\partial S_{xx}}{\partial x}\frac{c_g}{h}\eta_L\,dx = \int S_0 f'(x\text{-}c_g t)\frac{c_g}{h}\frac{-S_0/\rho}{gh-c_g^2} f(x\text{-}c_g t)dx \propto \int f(x\text{-}c_g t)f'(x\text{-}c_g t)dx$$
$$\propto \left[f^2(x\text{-}c_g t)\right]_{\text{start}}^{\text{end}} \tag{12}$$

which vanishes whenever S_{xx} has the same value at the start and at the end of the group.

However, in the scenario on the right of Figure 5, the long wave $\eta_L = \eta_{B,left} + \Delta\eta_B + \eta_{free}$ has the form

$$\eta_L = -(a_{B,left} + \Delta a_B)f(x - c_g t) + a_{free}^{forward} f(x - \sqrt{gh}t) + ... \tag{13}$$

where the sum of the incremental bound wave and the forward free wave is of the form $\Delta\eta_L = \Delta a_B(\sqrt{gh} - c_g)\delta_t f'(x - c_g t)$, corresponding to both having the shape f( ), with opposite sign and having moved the distance $(\sqrt{gh} - c_g)\delta_t$ apart, as per the derivation (9). This gives rise to a non-zero work contribution namely:

$$\dot{E} \approx \int \frac{\partial S_{xx}}{\partial x}\frac{c_g}{h}\Delta\eta_L\,dx = \int S_0 f'(x\text{-}c_g t)\frac{c_g}{h}\Delta a_L f'(x\text{-}c_g t)dx \propto \int f'^2(x\text{-}c_g t)dx > 0 \tag{14}$$

That is, the fact that the incremental long wave is composed of a bound part in the shape of $f(x - c_g t)$ and an (almost, since we neglect the backward free wave) equal and opposite forward free wave $f(x - \sqrt{gh}t)$, which combine to something of the form $f'(x - c_g t)$ explains the energy transfer to the long wave in the shoaling process qualitatively.

In terms of the curves in Figure 5, what has happened is that, the main part of the long wave, the depression, has been shifted backwards corresponding to the harmonic components having developed phase-shifts as noted by Elgar et al (1992) and Janssen et al (2003). This phase shift is necessary for transfer of energy from the short waves to the long wave. – Necessary for long wave growth.

Janssen et al (2003) stress the necessity for the phase-shifts to enable energy transfer and ensuing long-wave growth and develop a model for this phase-shift. However, their model development is somewhat obscure because they do not make explicit use of the free waves, which create the phase shifts as in Figure 5, in their description.

6. The time scale for surf-beat development

Longuet-Higgins & Stewart (1962) derived the quasi steady shoaling rate $\eta_B(h) \sim h^{-5/2}$, corresponding to taking the steady solution (3) to a shallower depth, while the short-waves shoaled according to green's law, i e, as $h^{-1/4}$. While LH&S did not talk about the growth process involving the free waves, which need time to move out of the way as described above, they did caution that the $h^{-5/2}$, growth rate would only be observed "given enough time". From the process described above and partly depicted in Figure 2, it is clear that the time scale for surf-beat development, ie, for the free waves moving out of the picture, is

$$\frac{L_L}{\sqrt{gh} - c_g} \approx \frac{L_L}{\sqrt{gh} - \sqrt{gh}[1 - \frac{1}{2}k_0 h + ..]} = \frac{2L_L}{\sqrt{gh}\,k_0 h} = \frac{L_L g^{1/2} T^2}{2\pi^2 h^{3/2}} \tag{15}$$

where L_L denotes the length of the long wave, same as the length of the short-wave group.

7. The shoaling rate is usually fastest for the higher-frequency long waves

The growth time-scale above corresponds to the 'shoaling rate' which was commented on by Elgar et al (1992) on the basis of the amplification of long-wave components travelling from h=13m to h=8m. They concluded on the basis of the data in Figure 6: *"The amplification is usually largest at the high frequency* *end of the infragravity band*"

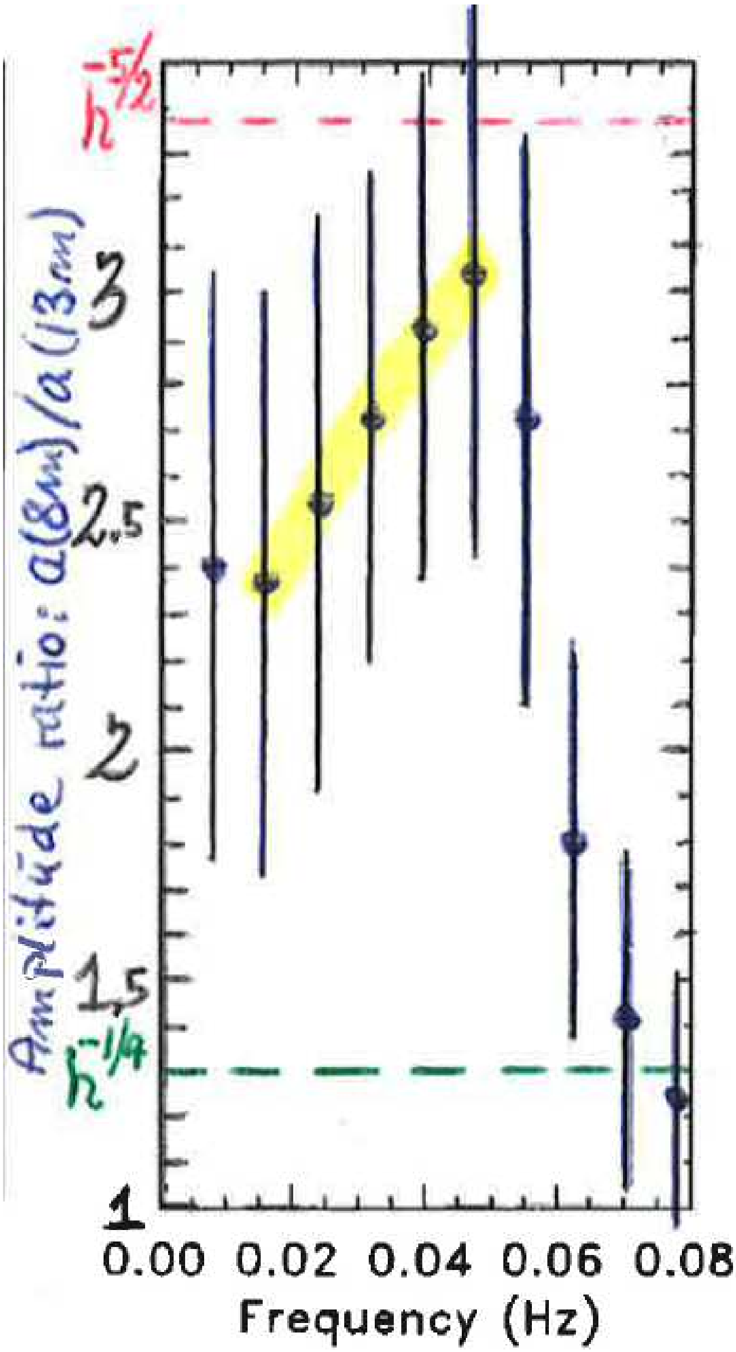

Figure 6:
Amplification rates for various wave bands going from h=13m to h=8m. Adapted from Elgar et al (1992). The longest components (f<0.02Hz) do not have time to develop fully, while the shortest 'infragravity components', $f \approx 0.05$Hz reach nearly full development corresponding to $h^{-5/2}$. Waves at 0.075Hz and above shoal according to Green's Law, ie, $h^{-1/4}$.

This corresponds to the fact that the resonant growth rate corresponding to (8) is proportional to the derivative of the shape function of the short-wave group.

8. Conclusions

The fact that, surf-beat usually travel through varying depths and therefore are not in a steady state, means that the details of their behavior cannot be understood simply through the steady solution of Longuet-Higgins & Stewart (1962). Analytical solutions, which account for the bed slope, are either restricted to bicromatic short-wave scenarios (Schaffer 1992) or become so complicated that the central, qualitative characteristic of the long-wave increments due to a depth change gets clouded for most readers (Janssen et al 2003) .

The central characteristic is that in general, and very obviously for a single wave pulse, continuity requires that the long-wave increment $\Delta\eta_L$ must represent zero net volume, ie, it cannot be simply a bowl-shaped increment to a pre-existing bowl. The complete solution to the transient problem over a horizontal bed shows how this can be handled. In this solution, every increment in the bound-wave solution $\Delta\eta_B$, is created together with a pair of co-located, free waves $\eta_{\text{free}}^{\text{forward}}$ and $\eta_{\text{free}}^{\text{backward}}$ so that, the co-located total gives no initial surface elevations and no extra energy or momentum. Subsequently, new surface elevations develop as the bound and free waves separate due to their different celerities ($\pm\sqrt{gh}$ versus c_g) and the total long-wave energy and momentum grows, as long as there is some overlap enabling energy transfer from the short-wave radiation stresses through non-linear terms of the form $\frac{\partial S_{xx}}{\partial x} U_L$. Laboratory experiments and numerical simulations, e g Lara et al (2011) show that the process of long-wave growth on a slope is qualitatively very similar to this process.

The fact that the development of long-wave amplitude is associated with $\eta_{\text{free}}^{\text{forward}}$ and $\Delta\eta_B$ separating due to the celerity difference $\sqrt{gh} - c_g$ explains that the time-scale for Long-wave amplitude growth is $\frac{L_L}{\sqrt{gh} - c_g}$ where L_L is the length of the long wave. Or, in terms of periods

$$\text{growth time scale} = \frac{L_L}{\sqrt{gh} - c_g} = \frac{T_L\sqrt{gh}}{\sqrt{gh} - c_g} = \frac{T_L}{\frac{1}{2}k_{0s}h + ...} \approx \frac{2T_L}{k_{0s}h} \qquad (16)$$

ie, the growth time scale is longer for longer surf-beats, or in other words, shorter surf-beats shoal faster than long ones. Hence, in the surf-beat shoaling scenario observed by Elgar et al (1992) the longest surf beats (0.02Hz<f<0.03Hz) did not have enough time to grow fully corresponding to the $h^{-5/2}$ depth dependence of (3) when travelling between their 13m and 8m stations. The shorter surf-beats (f~0.05Hz) on the other hand got much closer to complete development along the same path.

Alternatively, the faster growth of shorter surf-beats can be seen as resulting from the presence of the *x*-derivative of the forcing shape, f’, in the resonant and near-resonant solutions (8) and (9). This derivative is twice as big for half-length surf-beats all other things equal.

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